Section 8.9

Improper Integrals

Example 8.9.1. Compute $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$

Theorem 5.2.2. If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$. Further, the accumulation function

$$A(x) = \int_a^x f(t)\, dt$$

is continuous for all x in $[a, b]$ at which $f(x)$ is continuous.

So $f(x) = \dfrac{1}{x^2}$ is not necessarily integrable on $[-1, 1]$.

How do we properly investigate this integral? With limits.

Example 8.9.3. Compute $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}}$

Improper Integrals with Unbounded Integrand

1. If f is continuous on $(a, b]$ with $\lim_{x \to a^+} = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If f is continuous on $[a, b)$ with $\lim_{x \to b^-} = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If f is continuous on $[a, b]$ except at the interior point p where f is unbounded, then

$$\int_a^b f(x)\,dx = \lim_{c \to p^-} \int_a^c f(x)\,dx + \lim_{d \to p^+} \int_d^b f(x)\,dx.$$

4. If the limits in cases 1-3 exist, then the improper integrals *converge*; otherwise, they *diverge*.

As we are using improper integrals to deal with infinite limits, we would also be inclined to investigate integrals over intervals limiting to infinity.

Example 8.9.4. How do we make sense of the integral $\int_1^\infty \frac{1}{x^2} dx$?

Improper Integrals over Infinite Intervals

1. If f is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on $(a, \infty]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx.$$
where c is any real number.

4. If the limits in cases 1-3 exist, then the improper integrals *converge*; otherwise, they *diverge*.

Example 8.9.5. Compute $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$

Example 8.9.6. For the family of functions $f(x) = \dfrac{1}{x^p}$, for what values of p does $\displaystyle\int_1^\infty f(x)\,dx$ converge?

Surprising Geometry. Let R be the region bounded by the graph of $y = x^{-1}$ and the x-axis for $x \geq 1$.

1. What is the volume of the solid generated by revolving R about the x-axis?
2. What is the surface area of this solid?
3. What is the volume of the solid generated by revolving R about the y-axis?

There are some improper integrals for which we will be unable to compute the limit. Still, we may be able to prove convergence!

Theorem 8.9.7. Comparison Test for Improper Integrals. Suppose f and g are continuous on the interval $[a, \infty)$, with $f(x) \geq g(x) \geq 0$ for all $x \geq a$. Then

1. if $\displaystyle\int_a^\infty f(x)\,dx$ converges, so does $\displaystyle\int_a^\infty g(x)\,dx$.
2. if $\displaystyle\int_a^\infty g(x)\,dx$ diverges, so to does $\displaystyle\int_a^\infty f(x)\,dx$.

The theorem also holds for improper integrals on $(-\infty, a]$ and $(-\infty, \infty)$ appropriately reformulated.

Example 8.9.8. Does $\displaystyle\int_0^\infty e^{-x^2} dx$ converge or diverge?

Example 8.9.9. Compute the following improper integrals or determine if they diverge.

1. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{\sqrt[3]{x}}$

2. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$, for $a > 0$.

3. $\displaystyle\int_{0}^{1} \frac{x^3}{x^4-1} dx$

4. $\displaystyle\int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx$

5. $\int_0^1 \frac{dx}{x^p}$